A Compositional Game to Fairly Divide Homogeneous Cake

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0. Abstract

The central question in the game theory of cake-cutting is how to distribute a finite resource among players in a fair way. Most research has focused on how to do this for a heterogeneous cake in a situation where the players do not have access to each other's valuation function, but I argue that even sharing homogeneous cake can have interesting mechanism design. Here, I introduce a new game, based on the compositional structure of iterated cake-cutting, that in the case of a homogeneous cake has a Nash equilibrium where each of n players gets $1/n$ of the cake, using just $n-1$ cuts. Naive composition of the 'I cut you choose' rule leads to an exponentially unfair cake distribution so suffers from a high price of anarchy. This cost is completely eliminated by the BigPlayer rule.

1. Background

Proper etiquette demands that when two people share a piece of cake, one cuts the piece and the other chooses first. We call this game—usually referred to as *I cut you choose*—fair, because the Nash equilibrium of this game leaves both players happy, even if the cake contains different kinds of frosting and decoration. That is, after both get their piece, neither of the players wants to swap their piece for the other, a property of the equilibrium known as *envy-freeness*. Extending this beyond two players is non trivial. In fact, it can be proved that **for $n > 2$, there is no discrete cake-cutting game with an envy-free equilibrium that leaves each of n players with a connected piece of cake.**

I focus on the special case of a homogeneous cake, and show that an envy-free game can be constructed, using the compositional structure of iterated cake-cutting. The compositional structure of the game made it very easy to implement the game in the Open Game Engine, and the equilibrium is computationally verified.

This work highlights the power of compositional and categorical thinking in game theory, and shows that even non-experts can use it to develop and implement ideas.

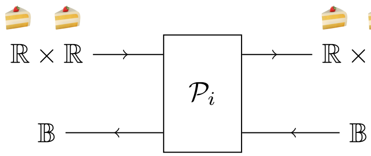
2. A homogeneous piece of cake

Dividing a homogeneous cake among n players is—in a sense—trivial, since there is an obvious partition that all players agree is fair (namely, everyone gets $1/n$), which the first player can immediately achieve with $n-1$ cuts.

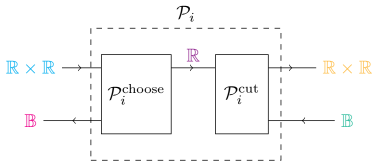
However, this is a very centralised game with uninteresting dynamics, and requires identifying a player that can exchange cake and information with everyone else. Rather, we want to focus on a game where fairness is achieved through a composition rule of 2-player *I cut you choose* games.

3. Compositional cake-cutting in the Open Game Engine

Imagine an iterated version of *I cut you choose*, where each player is offered two pieces, chooses one, cuts it, and presents their cut piece to the next player. This has a very nice (lens-like) open structure, where each player is presented with two pieces of cake, and outputs two pieces of cake, while receiving a response from the next player, and informing the previous player of their own choice:



Each player has to perform two actions: choose one of the incoming pieces, and cut the chosen piece. Diagrammatically:



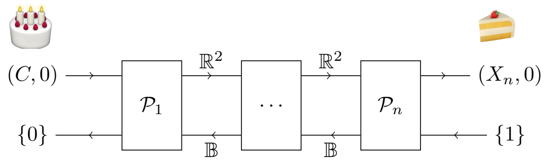
This is very naturally implemented in the Open Game Engine DSL (matching colours to the wires):

```
openCakeCuttingUnit playerName = [opengame|

inputs    : inputOffer ;
feedback  : playerResponse;
:----------------------------:
inputs    : inputOffer;
feedback  : playerResponse;
operation: CakeGame_choose playerName
outputs   : chosenPiece
returns   : ;

inputs    : chosenPiece;
feedback  : ;
operation: CakeGame_cut playerName ;
outputs   : newOffer ;
returns   : newResponse;
:----------------------------:
outputs   : newOffer    ;
returns   : newResponse;
|]
```

Such games can be composed into an n-player cake-cutting came by simply concatenating the n 1-player games and putting the composition into an appropriate context:



4. The BigPlayer rule eliminates the Price of Anarchy

Naive decentralisation like this means that every player is incentivised to choose the biggest piece, and cut their piece exactly in half. That means that the mth player get 2^{-m} of the cake (except the last player, who gets $2^{-(m-1)}$). In other words: vanilla compositional cake-cutting is **exponentially unfair**, which corresponds to a Gini-coefficient of (asymptotically) 1/2, which is about as unequal as the worst income inequalities in the world. Consider, instead, the following composition rule:

Definition 1 (The BigPlayer Rule).
Let $\{P_1, P_2, \ldots, P_n\}$ be the players in the n-player game of dividing a cake of size C. The BigPlayer rule then says the following:

1. P_1 cuts the cake in two, resulting in two pieces of sizes $(\alpha C, (1-\alpha)C)$, where $\alpha \in [0, 1]$.
2. P_2 chooses one of the two pieces.
If there are any players left who did not play yet:
3. Let the last cutter and the chooser be (P_a, P_b), respectively, and the size of their pieces (a, b), respectively. Then, let $P_{BP} = P_a$ if $a \geq b$, and $P_{BP} = P_b$ otherwise. PBP then has to cut their piece in two.
4. A player that did not play yet chooses one of P_{BP}'s pieces.
5. Move to 3 if there are any players that have not played yet.

This is also just iterated *I cut you choose*, but instead of simply composing the 1-player games, at each round the player who ends up with the biggest piece has to be the cutter in the next round. This game has an equilibrium in which every player gets 1/n of the cake, and obviously only involves n-1 cuts. The price of decentralisation is commonly quantified as the ratio of the maximum welfare among the set all possible outcomes S and among the set of equilibrium outcomes S_{eq}, also known as the **price of anarchy** (PoA). If we strive for equitable outcomes, we might take the welfare of a particular outcome s to be 1 minus the Gini-coefficient of cake distribution under s:

$$\text{PoA} = \frac{\max_{s\in S} W(s)}{\max_{s\in S_{eq}} W(s)} = \frac{\max_{s\in S} 1 - G(s)}{\max_{s\in S_{eq}} 1 - G(s)}$$

By ensuring that the fair distribution is an outcome in S_{eq}, the bigPlayer composition rule eliminates the price of anarchy in compositional cake-cutting.

Both vanilla composition, as well as BigPlayer composition have been implemented in the Open Game Engine, and different equilibria and profitable deviations are analysed. The code is available through a link in the paper:

5. Acknowledgements

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